SENIOR 'KANGAROO' MATHEMATICAL CHALLENGE

Friday 29th November 2013

Organised by the United Kingdom Mathematics Trust

The Senior Kangaroo paper allows students in the UK to test themselves on questions set for the best school-aged mathematicians from across Europe and beyond.

RULES AND GUIDELINES (to be read before starting):

1. Do not open the paper until the Invigilator tells you to do so.
2. Time allowed: **1 hour**.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. **Use B or HB pencil only** to complete your personal details and record your answers on the machine-readable Answer Sheet provided. **All answers are written using three digits, from 000 to 999.** For example, if you think the answer to a question is 42, write 042 at the top of the answer grid and then code your answer by putting solid black pencil lines through the 0, the 4 and the 2 beneath.

   Please note that the machine that reads your Answer Sheet will only see the solid black lines through the numbers beneath, not the written digits above. You must ensure that you code your answers or you will not receive any marks. There are further instructions and examples on the Answer Sheet.
5. The paper contains 20 questions. Five marks will be awarded for each correct answer. There is no penalty for giving an incorrect answer.
6. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

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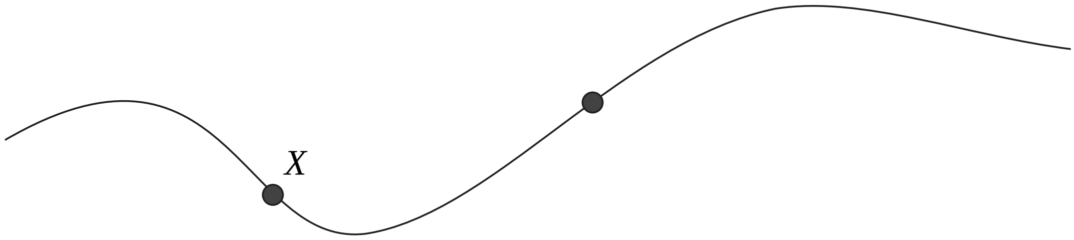
1. Adam, Bill and Carl have 30 sweets between them. Bill gives 5 sweets to Carl, Carl gives 4 sweets to Adam and Adam gives 2 sweets to Bill. Now each of them has the same number of sweets. How many sweets did Carl have initially?

2. An *i*-rectangle is defined to be a rectangle all of whose sides have integer length. Two *i*-rectangles are considered to be the same if they have the same side-lengths. The sum of the areas of all the different *i*-rectangles with perimeter 22 cm is A cm^2. What is the value of A?

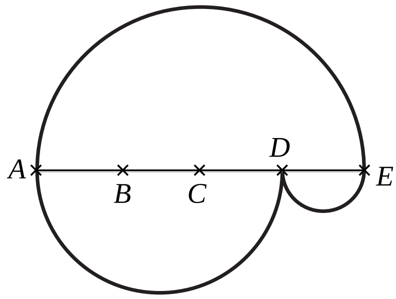
3. Some historians claim that the ancient Egyptians used a rope with two knots tied in it to construct a right-angled triangle by joining the two ends of the rope and taking the vertices of the triangle to be at the two knots and at the join. The length of the rope shown is 60 m and one of the knots is at X, which is 15 m from one end of the rope. How many metres from the other end of the rope should the second knot be placed to be able to create a right-angled triangle with the right angle at X?



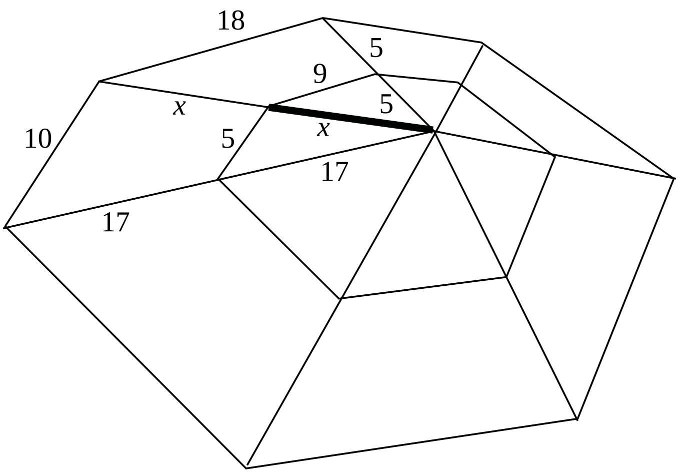
4. The height, width and length of a cube are multiplied by 2, 3 and 6 respectively to create a cuboid. The surface area of the cuboid is N times the surface area of the original cube. What is the value of N?

5. In a university admissions test, Dean gets exactly 10 of the first 15 questions correct. He then answers all the remaining questions correctly. Dean finds out he has answered 80% of all the questions correctly. How many questions are there on the test?

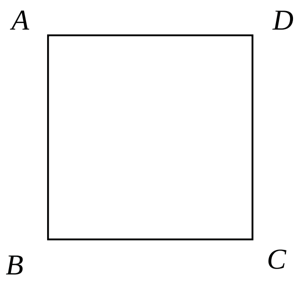
6. In the diagram, AE is divided into four equal parts and semicircles have been drawn with AE, AD and DE as diameters. This has created two new paths, an upper path and a lower path, from A to E. The ratio of the length of the upper path to the length of the lower path can be written as $a : b$ in its lowest terms. What is the value of $a + b$?



7. A mathematically skilful spider has spun a web and the lengths of some of the strands (which are all straight lines) are as shown in the diagram. It is known that x is an integer. What is the value of x?



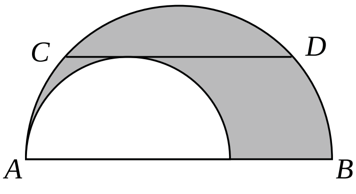
8. The square $ABCD$ has sides of length 1. All possible squares that share two vertices with $ABCD$ are drawn. The boundary of the region formed by the union of these squares is an irregular polygon. What is the area of this polygon?



9. In triangle ABC, angle B is 25% smaller than angle C and 50% larger than angle A. What is the size in degrees of angle B?

10. In the equation $2^{m+1} + 2^{m} = 3^{n+2} - 3^{n}$, m and n are integers. What is the value of m?

11. The diagram shows two semicircles. The chord CD of the larger semicircle is parallel to AB, and touches the smaller semicircle. The length of CD is 32 m. The area of the shaded region is $k\pi$ m^2. What is the value of k?



12. The sum of five consecutive integers is equal to the sum of the next three consecutive integers. What is the largest of these eight integers?

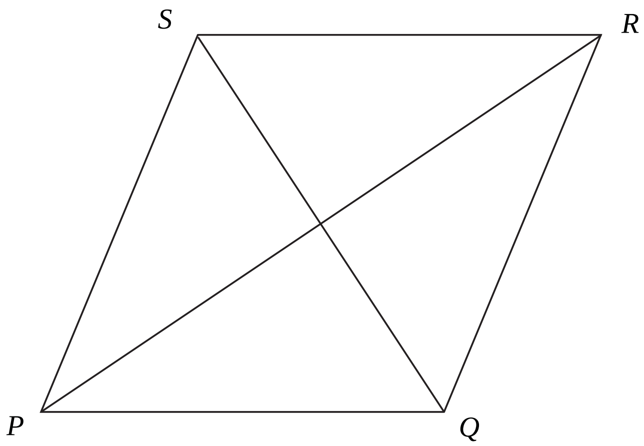
13. Zoe was born on her mother's 24th birthday so they share birthdays. Assuming they both live long lives, on how many birthdays will Zoe's age be a factor of her mother's age?

14. What is the largest three-digit integer that can be written in the form $n + \sqrt{n}$ where n is an integer?

15. How many integers a are there for which the roots of the quadratic equation $x^2 + ax + 2013 = 0$ are integers?

16. A sphere of radius 3 has its centre at the origin. How many points on the surface of the sphere have coordinates that are all integers?

17. The length of each side of the rhombus $PQRS$ is equal to the geometric mean of the lengths of its diagonals. What is the size in degrees of the obtuse angle PQR?



[The geometric mean of 2 values x_1 and x_2 is given by $\sqrt{x_1 x_2}$.]

18. How many of the first 2013 triangular numbers are multiples of 5?

19. The increasing sequence 1, 3, 4, 9, 10, 12, 13, … contains all the powers of 3 and all the numbers that can be written as the sum of two or more distinct powers of 3. What is the 70th number in the sequence?

20. Rachel and Nicky stand at either end of a straight track. They then run at constant (but different) speeds to the other end of the track, turn and run back to their original end at the same speed they ran before. On their first leg, they pass each other 20 m from one end of the track. When they are both on their return leg, they pass each other for a second time 10 m from the other end of the track. How many metres long is the track?

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SOLUTIONS

2013 Senior Kangaroo Solutions

1. **9** There are 30 sweets in total so, since the boys all finish with the same number of sweets, they must then have $30 \div 3 = 10$ sweets. Carl gains 5 sweets from Bill and gives 4 sweets to Adam so has a net gain of 1 sweet. Since Carl finishes with 10 sweets, he must start with $10 - 1 = 9$ sweets.

2. **110** The perimeter of each i-rectangle is 22 cm. Therefore, the sum of the length and the width is 11 cm. All the sides of the i-rectangle are whole numbers so the possible i-rectangles are 1×10 with area 10 cm^2, 2×9 with area 18 cm^2, 3×8 with area 24 cm^2, 4×7 with area 28 cm^2 and 5×6 with area 30 cm^2. Hence the sum of the areas of all possible i-rectangles is $10 + 18 + 24 + 28 + 30 = 110$ cm^2. Therefore the value of A is 110.

3. **25** Let the distance of the second knot from the other end of the rope be d m.

This part of the rope will become the hypotenuse of the right-angled triangle so, on applying Pythagoras' Theorem, we have the equation $d^2 = 15^2 + (45 - d)^2$. Now expand the brackets to get $d^2 = 225 + 2025 - 90d + d^2$. This simplifies to $90d = 2250$, which has solution $d = 25$.

Hence the second knot is 25 m from the other end of the rope.

4. **12** Let each side of the original cube have length x so that the cube has surface area $6x^2$. Then the cuboid has side-lengths $2x$, $3x$ and $6x$, so has surface area $2 \times (2x \times 3x + 2x \times 6x + 3x \times 6x) = 72x^2$.

Hence the value of N is $72x^2 \div 6x^2 = 12$.

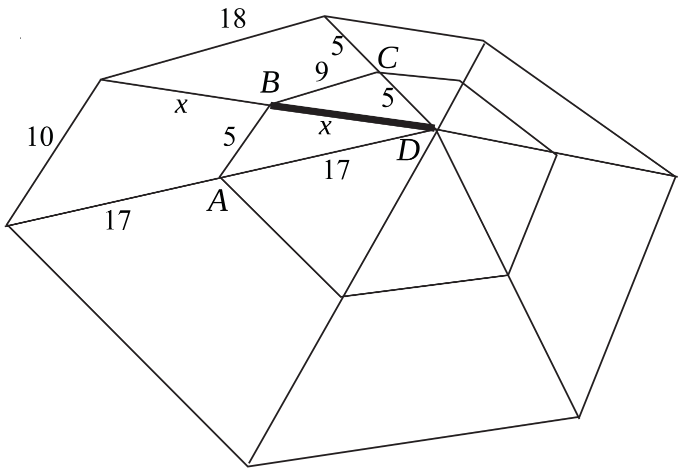
5. **25** Dean has answered 5 questions incorrectly so 5 questions must represent 20% of the questions. 20% is equivalent to $\frac{1}{5}$ so the total number of questions is $5 \times 5 = 25$.

6. **2** Let the length of AE be $4x$. Therefore, the lengths of AD and DE are $3x$ and x respectively.
The length of the upper path is $\frac{1}{2} \times \pi \times 4x = 2\pi x$.
The length of the lower path is $\frac{1}{2} \times \pi \times 3x + \frac{1}{2} \times \pi \times x = 2\pi x$.

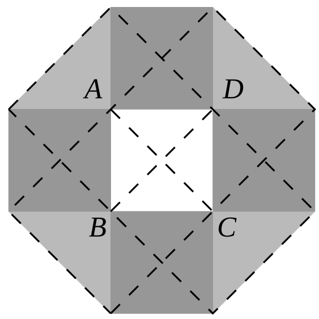
Therefore the ratio of the length of the upper path to the length of the lower path is 1 : 1. Hence the value of $a + b$ is 2.

7. **13** In any triangle, the length of the longest side is less than the sum of the lengths of the other two sides.

Apply this result (known as the triangle inequality) to the triangle BCD to obtain $x < 9 + 5$ or $x < 14$. In the same way, apply this result to triangle ABD to obtain $17 < x + 5$ or $x > 12$. But we are told that x is an integer and so $x = 13$.



8. **7** In addition to the original square, four squares can be drawn that share two adjacent vertices of the original square and a further four squares can be drawn that share two opposite vertices of the original square. The union of these squares creates the octagon as shown.



The octagon is made up of five squares of side 1 unit and four halves of a square of side 1 unit. Hence the area of the polygon is equal to $7 \times 1 \times 1 = 7$.

9. **60** Let the sizes of angles A, B and C be $a°$, $b°$ and $c°$ respectively.

From the question we have $b = 0.75 \times c$. Therefore $c = \frac{4}{3}b$.

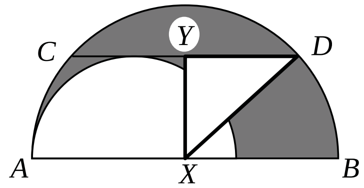
Similarly we have $b = 1.5 \times a$. Therefore $a = \frac{2}{3}b$.

Angles in a triangle add up to 180°, so that we have$180 = \frac{2}{3}b + b + \frac{4}{3}b$, which means that $180 = 3b$. It follows that $b = 60$.

10. **3** Factorise both sides of the equation to get $2^m(2^1 + 1) = 3^n(3^2 - 1)$. Thus we have $2^m \times 3 = 3^n \times 8$ which is equivalent to $2^{m-3} = 3^{n-1}$. Since 2 and 3 have no factors in common other than 1, a power of 2 cannot equal a power of 3 unless both powers are zero when both sides of the equation equal 1. Therefore we have $m - 3 = 0$ and $n - 1 = 0$. Hence the value of m is 3 (and the value of n is 1).

11. **128** Let the radii of the larger and smaller semicircles be R and r respectively. Then the shaded area as $\frac{1}{2} \times \pi R^2 - \frac{1}{2} \times \pi r^2 = \frac{1}{2}\pi(R^2 - r^2)$.

Let X be the centre of the larger semicircle and let Y be the midpoint of CD. Since CD is parallel to AB then $XY = r$ and $\angle XYD = 90°$. Apply Pythagoras' Theorem to triangle XYD to give $R^2 = r^2 + 16^2$ or $R^2 - r^2 = 256$. Therefore the shaded area is $\frac{1}{2}\pi(R^2 - r^2) = 128\pi$. Hence the value of k is 128.



12. **11** Let the smallest number be n. From the information in the question, we obtain the equation

$$n + n + 1 + n + 2 + n + 3 + n + 4 = n + 5 + n + 6 + n + 7.$$

Therefore we get $5n + 10 = 3n + 18$, which has solution $n = 4$.

Hence the largest number is $4 + 7 = 11$.

13. **8** Let Zoe be x years old. Therefore, her mother's age is $x + 24$ years old. Now x divides $x + 24$ if and only if x divides 24. The positive factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24 and so Zoe's age is a factor of her mother's age on 8 birthdays, when her mother's age will be 25, 26, 27, 28, 30, 32, 36 and 48.

14. **992** Since $n + \sqrt{n}$ is an integer, n is a square number. The square numbers near 1000 are $30^2 = 900$, $31^2 = 961$ and $32^2 = 1024$. Clearly, if $n = 32^2$ then $n + \sqrt{n}$ is greater than 1000, so this is not possible. However, if $n = 31^2$, then $n + \sqrt{n} = 961 + 31 = 992$, which is less than 1000. Hence the largest three-digit integer than can be written in the given form is 992.

15. **8** If the equation $x^2 + ax + 2013 = 0$ has integer solutions, then it can be written in the form $(x + b)(x + c) = 0$ for integers b and c. This means that $bc = 2013$.

As the prime factorisation of 2013 is $3 \times 11 \times 61$, so the possible factor pairs of 2013 are 1 and 2013, 3 and 671, 11 and 183 and 33 and 61. However, these only take into account the cases when both b and c are positive and four further pairs are possible if both b and c are negative. Thus there are 8 distinct values of a, namely ±2014, ±674, ±194 and ±94.

16. **30** Let the coordinates of a relevant point on the sphere be (x, y, z). By the three-dimensional version of Pythagoras' Theorem, we have $x^2 + y^2 + z^2 = 3^2$. The only solutions for which x, y and z are positive integers are (3, 0, 0) and (1, 2, 2) in some order.

There are $3 \times 2 = 6$ solutions based on the values (3, 0, 0) as the 3 can go in any of the three positions and be either positive or negative. Similarly there are $3 \times 2 \times 2 \times 2 = 24$ solutions based on the values (1, 2, 2) as the 1 can go in any of the three positions and all three of the values can independently be either positive or negative. This gives $6 + 24 = 30$ solutions in total.

17. **150** Let x be the length of a side of the rhombus and let a and b be the lengths of the two diagonals. The area of the rhombus is

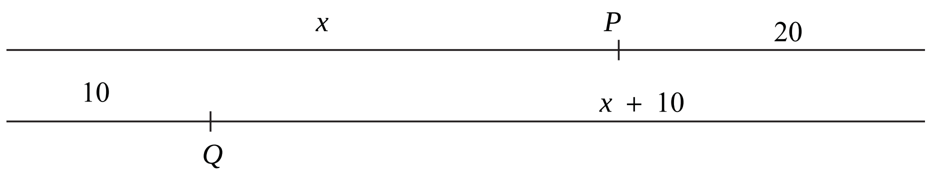
$$\text{area}\,\Delta QRS + \text{area}\,\Delta PQS = \tfrac{1}{2}x^2 \sin\angle SRQ + \tfrac{1}{2}x^2 \sin\angle SPQ.$$

Opposite angles in a rhombus are equal so this simplifies to $x^2 \sin\angle SRQ$. However, the area of a rhombus can also be calculated in a similar way to a kite, i.e. half the product of the diagonals. This gives the equation $x^2 \sin\angle SRQ = \tfrac{1}{2}ab$. From the question, we know that $x = \sqrt{ab}$ so $x^2 = ab$. Hence $\sin\angle SRQ = \tfrac{1}{2}$ and so $\angle SRQ = 30°$. Lines SR and PQ are parallel and so, using co-interior angles, $\angle PQR + \angle SRQ = 180°$. This means $\angle PQR = 150°$.

18. **804** The triangular numbers are given by the formula $T_n = \tfrac{1}{2}n(n + 1)$. T_n is a multiple of 5 if, and only if, one of n or $n + 1$ is also a multiple of 5. This means that two triangular numbers in every group of 5 consecutive triangular numbers will be a multiple of 5. None of the numbers 2011, 2012, 2013 or 2014 is a multiple of 5 and so none of T_{2011}, T_{2012} or T_{2013} is a multiple of 5 either. Hence the number of multiples of 5 in the first 2013 triangular numbers is $2 \times \frac{2010}{5} = 804$.

19. **741** Consider the n numbers $3^0, 3^1, 3^2, \ldots, 3^{n-1}$. Using at most one of each of these in a sum, the number of totals we can create is $2^n - 1$ as each of the n numbers can either be included or excluded from the sum but at least one number must be included so the choice of excluding all the numbers is discounted (and they are all distinct). For $n = 6$, this is 63 and so the 64th number in the sequence will be $3^6 = 729$. Then the 70th term is equal to the 64th term + 6th term = $729 + 12 = 741$.

20. **50** Let the distance Rachel runs before they first meet be x m. Let v_R and v_N be the respective speeds of Rachel and Nicky and let t_1 and t_2 be the times they take to get to their first and second passing points respectively (shown as P and Q on the diagram below).



As distance = speed × time, we have the following equations:

$$x = v_R t_1 \text{ and } 20 = v_N t_1;\ 2x + 30 = v_R t_2 \text{ and } x + 30 = v_N t_2$$

where the first and second pair give the distances travelled by Rachel and Nicky from the start to P and Q respectively.

Divide each equation in the first set by the corresponding equation in the second set to eliminate v_R and v_N to obtain $\dfrac{x}{2x + 30} = \dfrac{t_1}{t_2} = \dfrac{20}{x + 30}$. Now multiply both sides by the common denominator $(x + 30)(2x + 30)$ to obtain $x(x + 30) = 20(2x + 30)$. This simplifies to $x^2 + 30x = 40x + 600$, i.e. $x^2 - 10x - 600 = 0$. This factorises to $(x - 30)(x + 20) = 0$ with solutions $x = 30$ and $x = -20$. As x is measuring a distance, it must be positive so $x = 30$. Hence the length of the track is $30 + 20 = 50$ m.